

Mixed phases in $U(N)$ superconductivity

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We consider a general class of type-II superconductors which are described by N complex order parameters (OP), with an overall $U(N)$ symmetry for the energy functional. In the lowest energy state, each OP adopts its own particular lattice structure, making the spatial variation of the overall condensate very complex. We present detailed results for the case $N = 2$. We also treat the limit $N \rightarrow \infty$, which allows us to resolve a recent controversy, by showing that all previous studies are fundamentally flawed.

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I. INTRODUCTION

It has been appreciated for many years that enlarging the symmetry group of a given model can yield both new physical insights, and the possibility of exact calculations. A familiar example is the generalization of the Heisenberg model of ferromagnetism to the $O(N)$ model [1], which allows an exact analysis of the critical properties for $N \rightarrow \infty$. The large- N extension of the non-linear σ -model [2] has been similarly fruitful [3]. The idea has also been applied recently to non-equilibrium problems in turbulence [4], phase ordering [5] and interface roughening [6]. The implicit assumption involved is that the gross physical features are insensitive to the size of the symmetry group – an assumption which is largely borne out in the above applications.

The first main point of this article is to emphasize that there is a wide class of models whose physics does not change smoothly on enlarging the symmetry group – in the language of critical phenomena, these are systems whose ordered phase is *spatially inhomogeneous*. We shall concentrate exclusively on one such system; namely, the type-II superconductor, which condenses into the so-called mixed-state – a periodic array of flux lines [7]. However, non-trivial effects will be apparent in the other systems within this class (which may be fairly well characterized by having a wavelength selection mechanism in their ordered phase.)

It is our intention to highlight the main physical changes which occur when one generalizes the Landau-Ginzburg theory of superconductors to a $U(N)$ theory involving N complex order parameters (OP). The model is not new – in fact it has been studied many times in the past two decades or so [8–14] – however, the physics we shall discuss here appears to have been overlooked in previous studies [10–13]. Although many of our conclusions apply to arbitrary N , such is the richness of these systems, we shall discuss in detail only the cases $N = 2$ and $N \rightarrow \infty$.

Our use of the $U(N)$ symmetry is for convenience only, as it is the smoothest continuation of the model away from $N = 1$. Clearly for higher values of N there are an

increasingly large number of possible model Hamiltonians which one may construct from the N OP's. Within the field of unconventional superconductivity (as evidenced in the heavy-Fermion system UPt_3), the commonly adopted model is an $N = 2$ Ginzburg-Landau theory with a reduced symmetry [15]. The richness of the phase diagram is well-documented, with the Abrikosov lattice giving way to exotic structures such as fractionally quantized vortices, and flux textures. An application for $N = 3$ can be found in rotating He_3 . Near to the normal-A1 transition, the (18 component) OP may be reduced to three coupled complex fields [16], with the external angular velocity playing the role of an applied magnetic field. These examples serve to show that the strategy of enlarging the symmetry group of systems whose condensate is spatially inhomogeneous is a 'double-edged sword'. The advantage is that one finds an increasingly rich variety of phases as one increases N . The disadvantage is that one thereby loses immediate physical contact with the original model of interest ($N = 1$).

The remainder of this article is dedicated to exhibiting the strengths and weaknesses of the $U(N)$ model, in terms of its relation to conventional superconductivity. In section II, we define the model via the Landau-Ginzburg energy functional, and derive the mean field theory for arbitrary N . In section III we study in detail the mean-field solution for $N = 2$ and demonstrate that the Abrikosov state is unstable, giving way to a state of interlocked centered rectangular lattices. In section IV we review the large- N limit for this model. All previous studies [10–13] have assumed an Abrikosov state for one condensed mode, and treated the remaining $(N - 1)$ modes as massless. We show that this state is unstable, and that the true ordered state corresponds to a complicated structure in which each OP condenses into a periodic state, such that the overall condensate density is spatially constant. This leads to an identification of the transition with that of the $O(2N)$ model of ferromagnetism (in the large- N limit) in two fewer dimensions. An immediate consequence of this is that the transition is continuous and the lower critical dimension of the system is $d_l = 4$. We also discuss the subtleties of commuting

the limit $N \rightarrow \infty$ with the thermodynamic limit. It is of note that the large- N limit may be solved with no approximations – we adopt neither the London limit (gauge field fluctuations only) nor the lowest Landau level (LLL) approximation (OP fluctuations only). We end with our conclusions in section V.

II. FORMULATION OF THE MODEL AND MEAN FIELD THEORY

We define the model via the Landau-Ginzburg energy functional [17], which has the generic form (setting $\hbar = c = 1$)

$$\mathcal{H}[\psi_i, \mathbf{A}] = \int d^3x \left\{ \sum_i [(1/2m^*)|\mathbf{D}\psi_i|^2 + \alpha|\psi_i|^2] + (\beta/2) \sum_{i,j} |\psi_i|^2 |\psi_j|^2 + (1/2\mu_0)(\mathbf{H} - \nabla \times \mathbf{A})^2 \right\}, \quad (1)$$

where $\mathbf{D} = -i\nabla - e^*\mathbf{A}$, $\{\psi_i\}$ are a set of N complex order parameters, \mathbf{A} is the vector potential, \mathbf{H} is the external magnetic field, e^* is the effective charge, and m^* , α and β are phenomenological constants. It is useful to introduce dimensionless units [17], such that the energy functional (in the ordered phase) takes the form

$$\mathcal{H}[\psi_i, \mathbf{A}] = \int d^3x \left\{ \sum_i [|\mathbf{D}\psi_i|^2 - |\psi_i|^2] + (1/2) \sum_{i,j} |\psi_i|^2 |\psi_j|^2 + (\mathbf{H} - \nabla \times \mathbf{A})^2 \right\}, \quad (2)$$

where $\mathbf{D} = (1/i\kappa)\nabla - \mathbf{A}$. The parameter $\kappa > 1/\sqrt{2}$ is the ratio of the London penetration depth to the coherence length. In these units the critical external field $H_{c2} = \kappa$. When generalizing the above expressions to d dimensions, one should imagine the external field directed along $(d-2)$ dimensions, such that it is transverse to the (x, y) plane in which an Abrikosov-like lattice structure may form.

The simplest analysis one can make of this system is mean field theory which amounts to approximating the free energy by the saddle point value of \mathcal{H} . This may be obtained by solving explicitly for the classical fields which are solutions of the Landau-Ginzburg equations

$$\mathbf{D}^2\psi_i = \left(1 - \sum_j |\psi_j|^2\right)\psi_i \quad (3)$$

$$\nabla \times \mathbf{b} = (1/2)(\psi^*\mathbf{D}\psi + \psi\mathbf{D}^*\psi^*), \quad (4)$$

where $\mathbf{b} = \mathbf{H} - \nabla \times \mathbf{A}$ is the microscopic magnetic field.

Although it is not known how to solve these equations in general, an exact analysis is possible very close to the critical field H_{c2} . We shall briefly outline the solution in the case of $N = 1$, and then generalize the solution to

arbitrary N . Further details may be found in refs. [7,17]. Near to H_{c2} the OP is small, so that the first equation may be linearized, and at this order the gauge field is just given by $\mathbf{A}_0 = H_{c2}(0, x, \mathbf{0}_\perp)$. Thus, the OP may be expressed in terms of Landau levels [18]. In fact, the value of H_{c2} itself is determined by associating criticality with the LLL eigenvalue. For our purposes, the key point is that the first equation may be rearranged in the form

$$D_+ D_- \psi = 0, \quad (5)$$

where

$$D_\pm = \mathbf{D}_x \mp i\mathbf{D}_y, \quad (6)$$

and we have used the fact that the OP components are constant in the $(d-2)$ other directions. Thus the ground state ‘wavefunctions’ are characterized by the identity

$$D_- \psi = 0. \quad (7)$$

These functions are the LLL, and have a degeneracy proportional to the area of the system (transverse to the applied magnetic field). One is free to construct different sets of functions from the LLL’s. The most elegant is due to Eilenberger [19]. The Eilenberger basis functions span the space of the LLL, yet each basis function is a doubly periodic function (essentially a particular Abrikosov lattice).

Given the above properties of the OP near the upper critical field, it is possible to manipulate the second mean field equation such that the r.h.s. is the curl of a vector. In this way, one may integrate the equation to obtain Abrikosov’s first fundamental identity

$$b(\mathbf{r}) = H - (1/2\kappa)|\psi(\mathbf{r})|^2. \quad (8)$$

We see that the magnetic field is reduced within the sample by an amount proportional to the condensate density.

The scale of the condensate is set by Abrikosov’s second identity. This is a little more difficult to prove [17], but is essentially obtained by spatially averaging the first equation and calculating the first non-trivial corrections to eq.(7). The result is

$$(1/\kappa)(\kappa - H)\langle|\psi|^2\rangle + ((1/2\kappa^2) - 1)\langle|\psi|^4\rangle = 0, \quad (9)$$

where $\langle \dots \rangle$ denotes a spatial average.

Substituting this last relation into eq.(8) and spatially averaging, we find

$$B = \langle b \rangle = H - \frac{(\kappa - H)}{(2\kappa^2 - 1)\beta_A}, \quad (10)$$

where the Abrikosov ratio is given by

$$\beta_A = \langle|\psi|^4\rangle/\langle|\psi|^2\rangle^2 \geq 1. \quad (11)$$

One can translate this result in terms of the free energy, which one finds to be proportional to $-1/\beta_A$, indicating

that the system condenses into a state which minimizes β_A under the constraint that it is an Eilenberger function. This state turns out to be the well-known triangular lattice, for which $\beta_A \simeq 1.1596$.

Now we come to the general $U(N)$ case as described by eqs.(3) and (4). It is easy to show that most of the previous analysis follows through in a trivial way. Each OP component must satisfy $D_- \psi_i = 0$ (although it is important to remember that a given component may be zero). The first Abrikosov identity is generalized to

$$b(\mathbf{r}) = H - (1/2\kappa) \sum_i |\psi_i(\mathbf{r})|^2. \quad (12)$$

There also exist N relations setting the relative scales of the OP components. These have the form

$$(1/\kappa)(\kappa - H)\langle |\psi_i|^2 \rangle + ((1/2\kappa^2) - 1) \sum_j \langle |\psi_i|^2 |\psi_j|^2 \rangle = 0. \quad (13)$$

This allows us to spatially average eq.(12) to find the relation between the magnetic flux density and the magnetic field:

$$B = H - \frac{(\kappa - H)}{(2\kappa^2 - 1)\beta_g(N)}, \quad (14)$$

where the generalized Abrikosov ratio is given by

$$\beta_g(N) = \frac{\sum_{i,j} \langle |\psi_i|^2 |\psi_j|^2 \rangle}{(\sum_i \langle |\psi_i|^2 \rangle)^2} \geq 1. \quad (15)$$

Again, one may show that the minimal free energy is to be found by minimizing β_g , with the constraint that each condensed OP component is either zero, or an Eilenberger function. For general N one soon appreciates that such a minimization is non-trivial as the N components jostle for favorable positions and lattice structures within the ‘primitive cell’. For this reason we concentrate in the next section on the simplest case.

III. ANALYSIS OF THE CASE $N = 2$

We are now only concerned with two OP’s ψ_1 and ψ_2 . By reducing the $U(2)$ symmetry we could make contact with the two-OP models of $U\text{Pt}_3$ [15] for which a similar mean field analysis has been performed [20]. Our purpose here is to exemplify the physics of these multi-component systems – this will be of great benefit in our analysis of the large- N limit in the following section.

A few words concerning the Eilenberger basis [19] are required at this point. As mentioned before, the Eilenberger functions $\phi(\mathbf{r}|\mathbf{r}_0)$ satisfy $D_- \phi = 0$. The amplitude of ϕ is a doubly periodic function, with a fundamental cell scaled so as to have unit length in the x direction, and a

periodicity vector (ζ, η) , where η is fixed by the condition of flux quantization. The label \mathbf{r}_0 simply fixes the lattice position in space. The functions are normalized such that $\langle |\phi|^2 \rangle = 1$. The functions also have the symmetry property $\phi(\mathbf{r}|\mathbf{r}_0) = \exp[2\pi i(y_0/\eta)x]\phi(\mathbf{r} + \mathbf{r}_0|\mathbf{0})$. This allows one to recast the normalization condition as a completeness relation:

$$\int_{\text{cell}} d^2 r_0 |\phi(\mathbf{r}|\mathbf{r}_0)|^2 = \eta. \quad (16)$$

It is useful to define the integrals

$$I(\mathbf{r}_1, \mathbf{r}_2) = \int_{\text{cell}} d^2 r |\phi(\mathbf{r}|\mathbf{r}_1)|^2 |\phi(\mathbf{r}|\mathbf{r}_2)|^2. \quad (17)$$

Returning to our expression for β_g in the case $N = 2$, we have to minimize the expression

$$\beta_g(2) = \frac{\langle |\psi_1|^4 \rangle + \langle |\psi_2|^4 \rangle + 2\langle |\psi_1|^2 |\psi_2|^2 \rangle}{(\langle |\psi_1|^2 \rangle + \langle |\psi_2|^2 \rangle)^2}. \quad (18)$$

One may show that a minimum may only exist for either one of the OP’s being zero, or else for each OP to be equivalent (up to a relative spatial shift). In the former case the value of β_g will be the Abrikosov value ($\simeq 1.1596$). To investigate the latter case, we set $\psi_1 = A\phi(\mathbf{r}|\mathbf{0})$ and $\psi_2 = A\phi(\mathbf{r}|\mathbf{r}_0)$ which reduces the task of minimizing β_g to that of finding the lattice type and the relative shift \mathbf{r}_0 which minimize

$$\beta_g(2) = I(\mathbf{0}, \mathbf{0}) + I(\mathbf{0}, \mathbf{r}_0). \quad (19)$$

We have numerically investigated the above expression, restricting our search to lattices within the class of centered rectangular structures (which corresponds to choosing $\zeta = 1/2$). This class includes square and triangular lattices. Somewhat surprisingly the minimal energy solution corresponds to each OP component adopting a lattice with a primitive cell with opening angle θ equal to 15° . Figures 1 and 2 show contour plots of the two OP components. Regions of lighter shade correspond to higher values of the OP. It is more illuminating to plot the overall condensate $|\phi_1|^2 + |\phi_2|^2$, as is shown in Fig. 3. We then see a surprisingly rich structure. Maxima of the condensate correspond to regions of minimal magnetic flux and *vice versa*. The energy for this arrangement corresponds to the value $\beta_g(2) \simeq 1.0062$, which is very close to the lower bound of unity.

Our initial guess was that the OP components would adopt square lattices ($\theta = 45^\circ$), as this arrangement has a higher symmetry. We plot the components and the overall condensate in Figs. 4, 5 and 6 for this case. The energy of the square lattice arrangement is only fractionally higher with $\beta_g(2) \simeq 1.0075$.

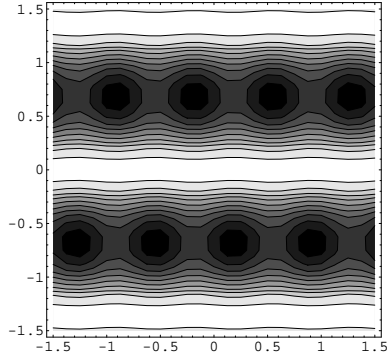


FIG. 1. $|\phi_1|^2$ for minimum energy lattice.

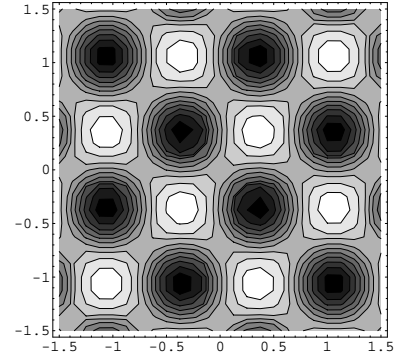


FIG. 4. $|\phi_1|^2$ for square lattice.

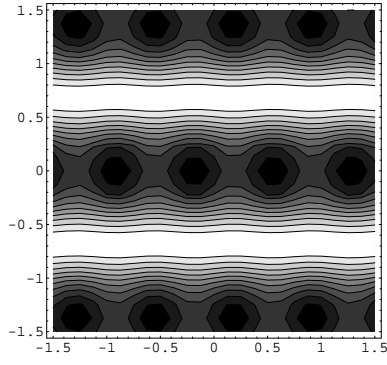


FIG. 2. $|\phi_2|^2$ for minimum energy lattice.

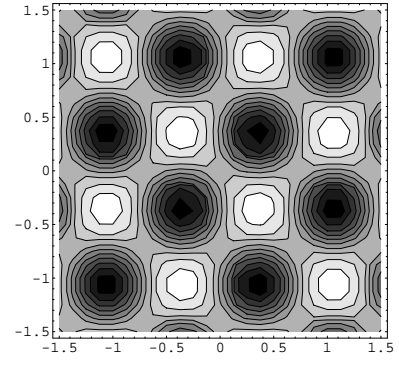


FIG. 5. $|\phi_2|^2$ for square lattice.

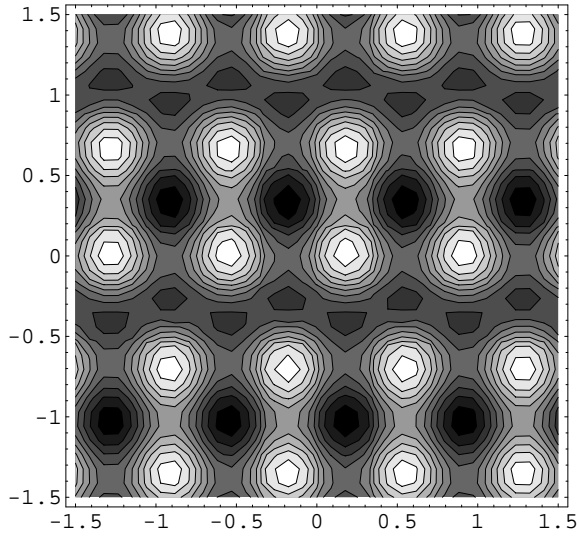


FIG. 3. Condensate $|\phi_1|^2 + |\phi_2|^2$ for minimum energy lattice.

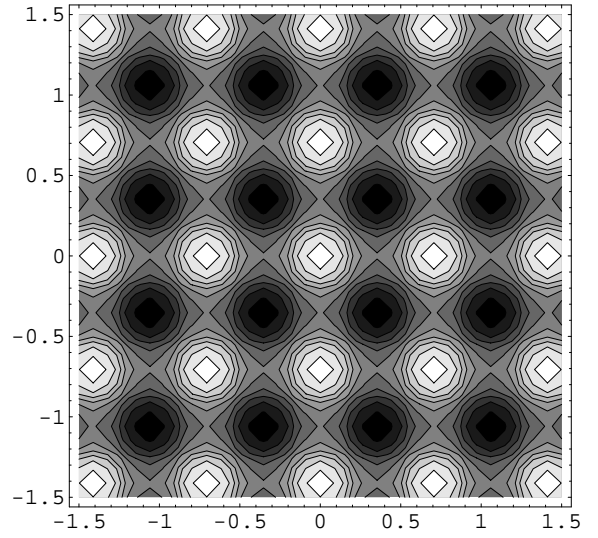


FIG. 6. Condensate $|\phi_1|^2 + |\phi_2|^2$ for square lattice.

In Fig.7 we show the generalized Abrikosov number as a function of angle. Interestingly, the triangular lattices correspond to maximal values of β_g . In principle, there may be an even lower energy OP arrangement corresponding to a lattice with an oblique primitive cell lying outside the class of centered rectangular structures. This is a large space within which to search for minima, and we have not pursued this possibility.

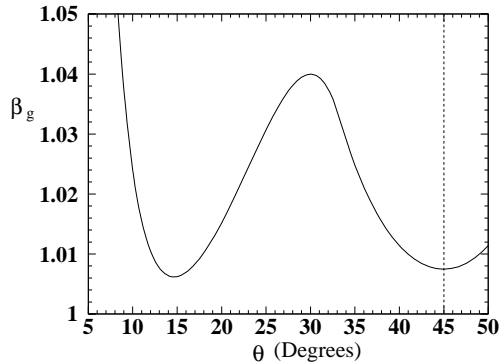


FIG. 7. The function $\beta_g(2)$ plotted against θ . The function is symmetric about 45° .

From this analysis, we see that in general the $U(N)$ models will condense into complicated structures in which each component adopts a particular lattice configuration, such that the overall condensate density is as smoothly varying as possible. The calculation of these structures becomes increasingly difficult as one increases N , and we shall desist from any explicit calculations. In the next section we turn to the most important part of the article – namely the explicit solution of the $U(N)$ model in the large- N limit.

IV. THE LARGE- N LIMIT

As hinted at in the Introduction, one of the more compelling reasons for generalizing models to higher symmetry groups is to allow exact solutions in the large- N limit. This has proven to be a very useful tool in many contexts [1,3–6]. In the present case, the large- N limit of the $U(N)$ model of type-II superconductors (in an external field) has been examined by several authors [10–13]. [One should draw a clear line between these calculations, and those concentrating on the zero field case [8], which is of interest in liquid crystals and also as an exemplification of a fluctuation induced first-order phase transition [21].] Originally studied in 1985 [10], it was found that below six dimensions, the mean-field continuous transition of Abrikosov becomes first-order – all the way down to a lower critical dimension of $d_l = 2$. This calculation

relied upon a proof by contradiction, as an explicit solution of the model is extremely difficult. An independent study [11] was made in 1995 using an Ansatz to solve the model. It was found that the transition was continuous below six dimensions, and that $d_l = 4$. This result was challenged [12] on the grounds that the Ansatz was physically unreasonable and that the original calculation [10] was definitely correct. A spirited response [13] was made, defending the Ansatz, but admitting that there was no obvious flaw in the original proof by contradiction [10]. This is quite an extraordinary situation regarding a model whose *raison d'être* is its solvability!

We shall resolve this state of confusion using a very simple physical idea that has emerged from the previous two sections. Each of the previous studies follows the conventional route of integrating out $(N - 1)$ of the OP components, regarding them as Goldstone modes. This is precisely what one does in the conventional $O(N)$ model of ferromagnetism for instance [1,22]. The reason it is done there is that the condensed OP is spatially homogeneous, and it makes good sense to globally rotate the OP's such that the condensate exists exclusively in one component (denoted as longitudinal), and to treat the transverse components as a source of massless fluctuations. Such a course of action is inadmissible in the present model of $U(N)$ superconductivity. Each component of the condensate is spatially inhomogeneous and no global rotation is possible to make $(N - 1)$ Goldstone modes. One has to treat all the modes as potentially condensed. For finite N this would lead to a totally intractable problem. However, for $N \rightarrow \infty$ we can take advantage of the completeness relation of Eilenberger functions to demonstrate that although each OP component is a doubly periodic function, the overall condensate is a constant. This drastically simplifies the analysis, and one may show that the solution is self-consistent when one takes into account both higher Landau levels, and gauge field fluctuations.

Before turning to the self-consistent large- N limit, we shall obtain the main idea by studying the limit of large N at mean field level. The task is to minimize $\beta_g(N)$ as given in eq.(15). We see that by choosing each component to lie in a different Eilenberger state, the sums over components tend (in the limit of large N) to integrals over the basis label \mathbf{r}_0 . We may then invoke the completeness relation given in eq.(16) which reduces these component sums to constants. Hence the spatial averages are trivial, and we see that β_g is saturated at its lower bound of unity.

It is a remarkable fact that this particular solution actually solves the mean field equations throughout the mixed phase, and not just in the vicinity of the H_{c2} line. One can see this straightforwardly: since $\sum_i |\psi_i|^2$ is a constant, the first mean-field equation is exactly solved by the Eilenberger functions. The higher Landau levels always remain hard modes as one decreases the temper-

ature, so never contribute to the condensed OP. At low enough temperatures, the Meissner transition will occur, and the condensate $\sum_i |\psi_i|^2$ will saturate at the value $2\kappa H$.

Two interesting points follow. In the previous section we found that even for two OP components, the condensed state was very smooth (in terms of the overall condensate). In that sense it is already very similar to the large- N limit in which the condensate is exactly constant. Thus we expect very similar quantitative physics as one increases N in the range $[2, \infty]$. Note also the huge degeneracy of lattice structures underlying this solution. Although each OP component is restricted to be an Eilenberger function with a particular shape of unit cell, there is no energetic selection of that unit cell for $N \rightarrow \infty$, since the completeness relation is independent of this. Presumably the ground-state OP configurations for large but finite N provide a means of determining the large- N configurations, by smoothly continuing $N \rightarrow \infty$.

The self-consistent large- N treatment is a non-trivial extension of the above calculation, as it takes into account fluctuations about mean-field theory (albeit in a rather crude fashion.) We shall find that the main characteristics of the mean field state are stable to these fluctuations for $d > d_l = 4$. For $4 < d < 6$ the fluctuations give rise to d -dependent exponents, which cross over to mean-field values above six dimensions. In fact, from the structure of the theory, one sees that the results are identical to those of the large- N limit of the $O(2N)$ model of ferromagnetism in two dimensions fewer. We can understand this as the correlations of the OP's are frozen in the (x, y) plane due to each OP component having formed a lattice state, so only transverse fluctuations (in the remaining $(d - 2)$ dimensions) can become critical.

[We note here that these results are similar to those of ref. [11], but we must stress that the physical condensed state is completely different in the two cases. In ref. [11] there is no explicit transverse scale as only one OP component is condensed, and is taken *ad hoc* to be constant (which is not even a solution of the saddle-point equations [12]); whereas in our solution, each OP component has condensed into a lattice state with its own magnetic length scale built in. Real thermal fluctuations about these two states will be of totally different natures.]

The large- N limit is often derived diagrammatically [1,22]. When one can describe the physics in terms of a longitudinal mode and $(N - 1)$ transverse modes, this approach is particularly transparent. However, in the present case we have N condensed modes, so it is better to use an alternative method; namely to introduce an auxiliary field χ (via a Hubbard-Stratonovich transformation) which will allow us to make the large- N limit explicit.

In the presence of fluctuations, we must take a step back from eq.(1) and consider the partition function $Z = \int \mathcal{D}\mathbf{A} \mathcal{D}\psi_i \exp[-\mathcal{H}']$, where

$$\mathcal{H}' = \mathcal{H} - \int d^d r \sum_i (J_i^* \psi_i + \text{c.c.}) . \quad (20)$$

The source terms J_i are added in order to derive the equation of state in the ordered phase. It will turn out that each J_i is proportional to an Eilenberger function.

Introducing the field χ allows us to rewrite the partition function in the form

$$Z[J] = \int \mathcal{D}\mathbf{A} \mathcal{D}\chi \mathcal{D}\psi_i \exp[-\mathcal{H}''], \quad (21)$$

where

$$\begin{aligned} \mathcal{H}'' = & \int d^d r \{ N\chi^2/2\beta + N(\mathbf{H} - \nabla \times \mathbf{A})^2 \\ & + \sum_i [(\alpha + i\chi)|\psi_i|^2 + (1/2m^*)|\mathbf{D}\psi_i|^2 - J_i^* \psi_i - J_i \psi_i^*] \} . \end{aligned} \quad (22)$$

We have scaled the magnetic field and the vector potential so as to extract a clean factor of N in the first two terms [10]. (This entails the rescalings $e^* \rightarrow e^*/\sqrt{N}$, and $\beta \rightarrow \beta/N$ for consistency).

As we have already indicated, the self-consistent large- N limit for this problem constitutes a formidable analytic challenge. In fact, it is not possible to solve the problem without resort to some external resource, whether it be an Ansatz, or a piece of physical insight. We shall utilize the latter, thanks to the lessons we have learned both in the $N = 2$ case, and also in the mean field analysis of the large- N limit. Just to reiterate, at mean field level the OP components each condense into an Eilenberger state such that the overall condensate is spatially homogeneous, and consequently the magnetic flux is spatially homogeneous also. To proceed we take the simplest possible line. Namely, that the self-consistent treatment retains these features of spatial homogeneity, but that the fluctuations renormalize the mass α , resulting in a shift of T_c . Our task is to show that this is a consistent solution of the problem. The alternative Ansatz is to condense only one OP component [10–13]. Rewriting the energy functional in terms of Eilenberger functions allows one to prove that such a state is energetically unstable [23].

To select the physically motivated condensed state we must choose the source terms to force each component into a (spatially shifted) Eilenberger function. Thus we write $J_i = u\phi_i = u\phi(\mathbf{r}|\mathbf{r}_i)$, where u is complex. The homogeneity of the magnetic field allows us to write $\mathbf{B} = B(0, 0, \hat{\mathbf{r}}_\perp)$. Also it is convenient to set $t = r + i\chi$. The saddle point value of χ is purely imaginary, such that the effective mass t is purely real. The energy function now takes the explicit form

$$\begin{aligned} \mathcal{H}'' = & \int d^d r \{ -N(t - r)^2/2\beta + N(H - B)^2 \\ & + \sum_i [t|\psi_i|^2 + (1/2m^*)|\mathbf{D}\psi_i|^2 - u\phi_i^* \psi_i - u\phi_i \psi_i^*] \} , \end{aligned} \quad (23)$$

where now $\mathbf{A} = B(0, x, \mathbf{0}_\perp)$. Each individual OP component is decoupled, and may be associated with a partition function $z(t, u) = \int \mathcal{D}\psi_i \exp(-f)$ where

$$f[\psi_i] = \int d^d r [\psi_i^* \hat{M} \psi_i - u \phi_i^* \psi_i - u \phi_i \psi_i^*] , \quad (24)$$

where $\hat{M} = -(1/2m^*)\mathbf{D}^2 + t$, which has eigenvalues

$$\lambda_{\mathbf{k},n} = \frac{k^2}{2m^*} + t + \left(n + \frac{1}{2}\right) \frac{e^* B}{m^*} , \quad (25)$$

where the momentum \mathbf{k} exists in the $(d-2)$ dimensions transverse to the (x, y) plane, and $n \in [0, \infty]$ labels the Landau levels.

We now write the condensed part of the OP explicitly, along with its fluctuation: $\psi_i = w\phi_i + \tilde{\psi}_i$. The prefactor w is chosen to ensure that the fluctuation piece $\tilde{\psi}_i$ has zero mean. Substituting this into eq.(24) and performing the integrals over the Eilenberger functions, one finds that the terms linear in $\tilde{\psi}_i$ vanish so long as one chooses $w = u/(t + e^* B/2m^*)$. In this case the energy functional for each component becomes

$$f[\psi_i] = -\frac{V|u|^2}{2(t + e^* B/2m^*)} + \int d^d r \tilde{\psi}_i^* \hat{M} \tilde{\psi}_i , \quad (26)$$

where V is the volume of the system.

Returning now to the energy functional, we have from eqs.(23) and (26), along with assumed spatial constancy of χ and B ,

$$\begin{aligned} \mathcal{H}'' = & -NV \left[\frac{(t-r)^2}{2\beta} - (H-B)^2 + \frac{|u|^2}{2(t + e^* B/2m^*)} \right] \\ & + \int d^d r \tilde{\psi}_i^* \hat{M} \tilde{\psi}_i . \end{aligned} \quad (27)$$

The integrals over the fluctuation fields $\{\tilde{\psi}_i\}$ are easily done, and we may re-exponentiate the resulting determinant to give the final result

$$\begin{aligned} \mathcal{H}'' = & -NV \left[\frac{(t-r)^2}{2\beta} - (H-B)^2 + \frac{|u|^2}{2(t + e^* B/2m^*)} \right. \\ & \left. - (D_L/A) \int \frac{d^{d-2}k}{(2\pi)^{d-2}} \sum_n \log(\lambda_{\mathbf{k},n}) \right] , \end{aligned} \quad (28)$$

where D_L is the Landau degeneracy of each level, which is equal to $e^* BA/2\pi$, where A is the area of the system in the (x, y) plane.

Now that we have a clean factor of N throughout, we may use steepest descents to determine the self-consistent values of the auxiliary variable t , and also the magnetic field B . Differentiating \mathcal{H}'' with respect to t yields

$$\begin{aligned} \frac{(t-r)}{\beta} - \frac{|u|^2}{2(t + e^* B/2m^*)^2} \\ - \frac{e^* B}{2\pi} \int \frac{d^{d-2}k}{(2\pi)^{d-2}} \sum_n \frac{1}{\lambda_{\mathbf{k},n}} = 0 . \end{aligned} \quad (29)$$

It is convenient to define $\xi^{-2} = t + e^* B/2m^*$, since ξ can be seen to play the role of the correlation length of the system. The relation between w and u then assumes the form $w = u\xi^2$. One may then rewrite eq.(29) as

$$\begin{aligned} \xi^{-2} = & r + e^* B/2m^* + \beta|w|^2/2 \\ & + \beta \frac{e^* B}{2\pi} \int \frac{d^{d-2}k}{(2\pi)^{d-2}} \sum_n \frac{1}{[k^2/2m^* + ne^* B/m^* + \xi^{-2}]} . \end{aligned} \quad (30)$$

This equation encapsulates most of the information about the phase transition. Above the transition, we can set the condensate ‘amplitude’ w to zero in (30). The resulting equation defines the renormalized critical temperature T_c through the defining condition of criticality $\xi \rightarrow \infty$. At this point the bare quantity $r = T - T_c^0$ is equal to $T_c - T_c^0$. We therefore have the explicit shift as

$$\begin{aligned} T_c = & T_c^0 - e^* B/2m^* \\ & - \beta \frac{e^* B}{2\pi} \int \frac{d^{d-2}k}{(2\pi)^{d-2}} \sum_n \frac{1}{[k^2/2m^* + ne^* B/m^*]} . \end{aligned} \quad (31)$$

As usual, the fluctuations drive the critical temperature to a lower value. The T_c shift diverges for $d < 4$, suggesting the identification of the lower critical dimension as $d_l = 4$.

On the low temperature side of the transition we can remove the source field u , which leaves a non-zero condensate w only if the correlation length is infinite. We therefore have from (30), an equation for the condensate amplitude:

$$\begin{aligned} 0 = & r + e^* B/2m^* + \beta|w|^2/2 \\ & + \beta \frac{e^* B}{2\pi} \int \frac{d^{d-2}k}{(2\pi)^{d-2}} \sum_n \frac{1}{[k^2/2m^* + ne^* B/m^*]} . \end{aligned} \quad (32)$$

Comparing eqs.(31) and (32) we find the exact relation $|w|^2 = 2(T_c - T)/\beta$ which immediately yields the OP exponent $\beta = 1/2$, and self-consistently confirms the existence of a continuous transition.

We can also identify the correlation length exponent by examining (30) as $T \searrow T_c$. Eliminating the bare critical temperature from (30) using (31), and evaluating the resulting integral for large ξ leads to the expression

$$c_1 \xi^{-2} + \beta e^* B c_2 \xi^{4-d} = T - T_c . \quad (33)$$

The constant c_1 has a contribution from all Landau levels bar the lowest. The fluctuation dominated term $\sim \xi^{4-d}$ and arises solely from the LLL. As the correlation length diverges for $T \searrow T_c$, we see that the first term on the left hand side dominates for $d > 6$, whereas the second term dominates for $4 < d < 6$. This leads to the result $\xi \sim (T - T_c)^{-\nu}$, with $\nu = 1/2$ for $d > 6$ (the mean field result), and $\nu = 1/(d-4)$ for $4 < d < 6$ (confirming $d_l = 4$).

These results are identical to those obtained for the $O(2N)$ model of ferromagnetism, but in two fewer dimensions. This may be understood from examination of the self-consistent relation for ξ given in (30). Apart from the sum over Landau levels, this equation is exactly that which would be obtained for a $(d-2)$ -dimensional $O(2N)$ model. The critical modes only exist in the $(d-2)$ dimensions transverse to the Landau levels. The modes in the (x, y) plane contain the frozen length scale associated with the formation of the underlying lattice structure of the OP's.

In the above expressions, we have left the value of the magnetic field B undetermined. However, this is given self-consistently by minimizing the energy functional in (28) with respect to B . We shall not write the expression explicitly, but it is noteworthy that the integral appearing from the fluctuations is strongly divergent and must be regularized by introducing some microscopic cut-off procedure (for $d > 4$), such as adding higher derivative terms not present in our original Landau-Ginzburg energy functional.

As a final remark in this section, we should point out that these results are insensitive to the order in which one takes the thermodynamic limit, and the limit $N \rightarrow \infty$. The above analysis has implicitly assumed the thermodynamic limit, which is the correct 'physical' choice. However, had one taken the thermodynamic limit second, by fixing the number of vortices and then taking $N \rightarrow \infty$, the calculation would have proceeded as before with one difference. The fluctuations in this case would originate overwhelmingly from the transverse modes, since the number of distinct longitudinal modes is limited to the number of vortices. However the completeness relation still holds, and the condensed state may be taken as spatially homogeneous, thereby enabling a calculation in the same spirit as that above.

V. CONCLUSIONS

The main point of this article is that special care must be taken when expanding the symmetry group of systems with spatially varying structures. We have concentrated on one class of such systems, namely type-II superconductors in an external magnetic field. As mentioned in the Introduction, there are applications of such models to heavy-Fermion superconductors, and also to rotating superfluid He_3 . We have found a number of interesting results connected with N -component superconductors, whose free energy functional maintains a $U(N)$ symmetry. In general we have seen that these systems adopt low-temperature configurations in which many OP components contribute by condensing into periodic structures – leading to a very rich structure for the overall condensate (and hence the magnetic flux).

We have examined the mean-field theory for the case $N = 2$ in some detail. The two OP components were found to condense into centered rectangular structures, with an opening angle of 15° . The two structures are shifted relative to one another in such a way that the overall condensate $|\phi_1|^2 + |\phi_2|^2$ has a surprisingly rich structure, as shown in fig.3. The generalized Abrikosov ratio for this configuration is $\beta_g(2) \simeq 1.0062$, almost saturating the lower bound of unity. Although systems with higher values of N will adopt ever more complex structures, we were able to show in section IV that the mean-field theory in the limit $N \rightarrow \infty$ has a simplifying feature, due to the completeness relation of the periodic Eilenberger functions. Each OP component adopts an Eilenberger function, but the overall condensate has no traces of the periodicity, and is spatially a constant. This structure saturates the lower bound of $\beta_g(\infty)$.

In the remainder of section IV we examined the large- N limit in more detail, by considering a treatment which includes fluctuations self-consistently. This calculation has been attempted several times in the past [10–13], but the previous authors have always assumed that the system allows only one OP component to condense, which we have seen is generically false. Our main finding is that for $d > 4$, fluctuations do not disturb the main characteristics found in mean-field theory – namely a continuous transition into a spatially homogeneous condensate, composed of infinitely many OP components having condensed into Eilenberger functions. [For $d < 4$, the mixed phase is destroyed entirely.] The system is found to have the critical properties of the $O(2N)$ model of ferromagnetism [1,22], but in two fewer dimensions. Thus, exponents maintain their mean-field values above $d = 6$, but take the values $\beta = 1/2$ and $\nu = 1/(d-4)$ for $4 < d < 6$. The solution we have found does not rely upon making the LLL approximation, or upon neglecting gauge field fluctuations.

In this paper we have found that within mean-field theory, and also in the large- N limit, the $U(N)$ model undergoes a continuous transition from the normal to the mixed phase. This is in contradiction to the results of some past works [9,10,12], the latter two of which contain errors of principle. However, one may find a precedent for a continuous transition in our recent study [14], in which a functional renormalization group (FRG) approach was applied to the $U(N)$ model via an expansion in $\epsilon = 6 - d$. In fact, the FRG study also predicted a mapping from the $U(N)$ model to the $O(2N)$ model of ferromagnetism in two fewer dimensions, for $N \geq 2$. It would be interesting to probe this relationship further by extending the present self-consistent analysis of the $U(N)$ model to finite N . One of the more sophisticated means of achieving this would be via the use of the parquet approximation [24], which includes corrections far beyond those of $O(1/N)$.

Finally we would like to draw the reader's attention to

the fact that in the exactly solvable large- N limit, the mechanism for the transition is the growing (phase) coherence in the direction of the applied field, as the temperature is lowered. A theory of the “melting” transition seen in high-temperature superconductors ($N = 1$) has been recently given by one of us [25], based on the idea that the apparent melting is just a consequence of crossover effects, when this phase coherence length scale (which is very rapidly growing in three dimensions), becomes comparable to the dimensions of the system.

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